

Three loop $\overline{\text{MS}}$ anomalous dimension for renormalizable gauge invariant non-local gluon mass operator in QCD

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Abstract. The three loop anomalous dimension for the gauge invariant, renormalizable, non-local mass operator for a gluon is computed in the $\overline{\text{MS}}$ scheme. In addition the anomalous dimensions of the associated localizing ghost fields are also deduced at the same order and it is shown that the three loop QCD β -function correctly emerges from the gluon localizing ghost vertex renormalization.

In non-abelian gauge theories the vector bosons responsible for carrying the quanta of force are regarded as massless particles unless there is a spontaneous symmetry breaking. Expressed another way there is no gauge invariant local mass operator for gluons in quantum chromodynamics (QCD). Whilst it is possible to have BRST invariant masses, such as that which occurs in the Curci-Ferrari model, [1], the inclusion of such local mass operators all suffer from the disadvantage of leading to non-unitary theories, [2, 3]. Hence they have no predictive power in relation to S -matrix elements. By contrast, there has been an explosion of interest in recent years in studying the infrared dynamics of Yang-Mills theories in the infrared limit in the Landau gauge using lattice techniques, Dyson-Schwinger equation methods and other more formal approaches. One of the main quantities which is analysed is the gluon propagator and it is widely acknowledged that it does not satisfy the usual perturbative form of a massless propagator of an unconfined field. Instead it is generally fair to say that the gluon propagator, as measured on the lattice and other methods, has a behaviour which is not inconsistent with the gluon having an effective mass of some sort. Whether this effective mass is due to screening, dynamically generated, derived from say Gribov issues, due to vortex condensation or another mechanism has not yet been definitively answered. However, if it is to be explained theoretically then one is forced into studying extensions of the Yang-Mills or QCD Lagrangians which have a concrete gluon mass term of some sort or one where a mass operator condenses. Clearly to do this in a gauge invariant way would appear impossible as the obvious mass operator, $\frac{1}{2}(A_\mu^a)^2$, breaks gauge symmetry despite being renormalizable, [1], where A_μ^a is the gluon field. However, if one sacrifices the restriction to *local* operators then it is possible to have several gauge invariant gluon mass terms. In essence there are two types.

The first, originally introduced in [4] in three dimensions, has been examined in four dimensions in [5, 6] where it was shown to be renormalizable. Indeed its two loop $\overline{\text{MS}}$ anomalous dimension was computed in [6] and shown to be independent of the gauge fixing parameter of a linear covariant gauge. The key to demonstrating renormalizability and allowing one to calculate in a systematic way was the fact that the Lagrangian involving the operator itself could be written in terms of local fields additional to the usual gluon, quark and Faddeev-Popov ghost fields. These extra (infrared) fields do not affect the usual ultraviolet properties of the original non-abelian gauge theory, [5, 6]. Therefore, for example, the β -function of [7] is unchanged. The other type of mass operator is in effect the Stueckelberg term but written as $\min_{\{U\}} \int d^4x (A_\mu^{aU})^2$ where U is an element of the gauge group. It has been studied in the massive gauge invariant model in [8, 9] and is central to a vortex interpretation of confinement. Although also being non-local it suffers from the calculational drawback of being non-renormalizable. Though its one loop anomalous dimension was computed in [10] for arbitrary linear covariant gauge and shown to be independent of the gauge parameter. Part of that calculation rested on the fact that the massive gauge invariant operator $\min_{\{U\}} \int d^4x (A_\mu^{aU})^2$ has the renormalizable non-local mass operator of [5, 6] as its first term in a gluon leg expansion of the operator in terms of gauge invariant operators, [11]. Therefore the localization of the previous non-local operator into the original Yang-Mills fields plus localizing ghost fields provided a useful calculational shortcut. From another point of view the non-local operator $\min_{\{U\}} \int d^4x (A_\mu^{aU})^2$ can be viewed as a method of gauge fixing QCD in a more concrete fashion as noted in [11, 12, 13, 14]. This is because that gauge fixing operator is gauge invariant and thus avoids the Gribov problem, [15], which plagues the more widely used Landau gauge in the present intense activity into the infrared structure.

From a theoretical point of view one would ultimately like to have a Lagrangian based method of studying effective gluon mass which emerges in the current picture and which is renormalizable. Moreover, as performing calculations is essential to understanding such low energy problems, we focus here on providing the anomalous dimensions of the non-local mass

operator of [5, 6] to *three* loops in the $\overline{\text{MS}}$ scheme. This is far from being a trivial exercise which is due in part to the presence of the additional fields but also because of the generation of a set of quartic interactions. As was shown in [5, 6] these are essential to preserving multiplicative renormalizability. Therefore, we also report on the renormalization of the fields themselves at three loops. Indeed as an example of where such three loop results are necessary we note that in [16, 17] the problem of the dynamical generation of a gluon mass was studied in the Landau gauge based on the local operator $\frac{1}{2}(A_\mu^a)^2$. Briefly, the two loop effective potential for this operator was computed for N_f massless quarks using the local composite operator (LCO) formalism, [18, 19]. Knowledge of this potential allows one to show that the energetically favoured vacuum is one where the operator condenses and therefore dynamically generates a gluon mass. One peculiar feature of the LCO formalism, however, is that to have the full two loop potential one needs the operator's anomalous dimension at *three* loops, [16, 17, 18, 19]. Whilst the results successfully demonstrated operator condensation which was stable to loop corrections, [16, 17], it suffers from one obvious drawback and that is that the calculation was restricted to a specific gauge. It would be more appropriate to study the extended operator considered here since it is gauge invariant. Indeed this is one of our motivations for this article. However, as will be evident from what we present, we believe the determination of this three loop anomalous dimension for the non-local operator is sufficiently interesting in its own right to present it separate from an LCO computation.

We begin by recalling the full form of the Lagrangian of [5, 6]. It is

$$\begin{aligned}
L = & -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu^a)^2 - \bar{c}^a \partial^\mu D_\mu c^a + i\bar{\psi}^{iI} \not{D}\psi^{iI} \\
& + \frac{1}{4} \left(\bar{B}_{\mu\nu}^a D_\sigma^{ab} D^{bc\sigma} B^{c\mu\nu} - \bar{H}_{\mu\nu}^a D_\sigma^{ab} D^{bc\sigma} H^{c\mu\nu} \right) + \frac{im}{4} (B_{\mu\nu}^a - \bar{B}_{\mu\nu}^a) G^{a\mu\nu} \\
& + \frac{1}{16} \lambda^{abcd} \left(\bar{B}_{\mu\nu}^a B^{b\mu\nu} - \bar{H}_{\mu\nu}^a H^{b\mu\nu} \right) \left(\bar{B}_{\sigma\rho}^c B^{d\sigma\rho} - \bar{H}_{\sigma\rho}^c H^{d\sigma\rho} \right)
\end{aligned} \tag{1}$$

where α is the linear covariant gauge fixing parameter, c^a is the Faddeev-Popov ghost, ψ^{iI} is the (massless) quark, $B_{\mu\nu}^a$ and $H_{\mu\nu}^a$ are the localizing ghosts where the latter are anticommuting and m is the gluon mass. For completeness the index ranges are $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ where N_F and N_A are the respective dimensions of the fundamental and adjoint representations and N_f is the number of quarks. The covariant derivative, involving the coupling constant g , is denoted by D_μ^{ab} and $G_{\mu\nu}^a$ is the field strength. The quantities λ^{abcd} are the quartic couplings necessary for multiplicative renormalizability and satisfy the symmetry properties

$$\lambda^{abcd} = \lambda^{bacd} = \lambda^{abdc} = \lambda^{cdab} . \tag{2}$$

They are not to be confused with the specific rank 4 invariant tensors, such as the totally symmetric tensor d_F^{abcd} of [20], which can be built from the structure functions, f^{abc} , or the colour group generators, T^a . In addition, since the Lagrangian is colour symmetric the quartic couplings satisfy a Jacobi style identity, [5, 6], which is

$$f^{apq} \lambda^{pbcd} + f^{bpq} \lambda^{apcd} + f^{cpq} \lambda^{abpd} + f^{cpq} \lambda^{abcp} = 0 . \tag{3}$$

In (1) we have ignored the masses of the $\{B_{\mu\nu}^a, \bar{B}_{\mu\nu}^a, H_{\mu\nu}^a, \bar{H}_{\mu\nu}^a\}$ sector since they will play no role in the present calculation. Finally, we note that (1) is the localized version of the Lagrangian with the explicit non-local mass operator, [5],

$$L = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu^a)^2 - \bar{c}^a \partial^\mu D_\mu c^a + i\bar{\psi}^{iI} \not{D}\psi^{iI} - \frac{m^2}{4} G_{\mu\nu}^a \left(\frac{1}{D^2} \right)^{ab} G^{b\mu\nu} \tag{4}$$

where D^2 is the square of the covariant derivative.

Clearly with the additional ghost fields and coupling one has to ensure that the gluon, ghost and quark anomalous dimensions as well as the usual β -function remain independent of λ^{abcd} . This has been verified at two loops in [5] and [6]. Therefore, here we will compute the former anomalous dimensions to three loops as well as those for $B_{\mu\nu}^a$ and $H_{\mu\nu}^a$. The latter will be λ^{abcd} -dependent. To deduce the anomalous dimension of the mass m or equivalently the mass operator anomalous dimension we will renormalize the dimension three gauge invariant operator \mathcal{O} where

$$\mathcal{O} = \frac{1}{4} (B_{\mu\nu}^a - \bar{B}^{a\mu\nu}) G^{a\mu\nu} \quad (5)$$

by inserting it into a gluon $B_{\mu\nu}^a$ two point function. The advantage of this approach is that one can split the free and interaction Lagrangian in such a way that the operator is in the latter and not the former. If it were included in the free part then we would have the huge (and unnecessary) computational task of calculating with massive propagators which would require the inclusion of the masses of $\{B_{\mu\nu}^a, \bar{B}_{\mu\nu}^a, H_{\mu\nu}^a, \bar{H}_{\mu\nu}^a\}$. (The explicit mass terms are given in [5, 6].) This would be an intractable proposition. Instead treating the operator as an insertion means that all fields remain massless and one also avoids the mixing of masses which occurs in the full quadratic sector of such a Lagrangian split, aside from the additional complications from the $B_{\mu\nu}^a$ and $H_{\mu\nu}^a$ masses. More crucially with massless fields one can employ the MINCER algorithm, [21], which has been encoded, [22], in the symbolic manipulation language FORM, [23]. The MINCER procedure applies to massless three loop 2-point functions, [21], and performs the computation in dimensional regularization in $d = 4 - 2\epsilon$ dimensions where ϵ is the regularizing parameter. Such a high loop order calculation can clearly only be performed via automatic Feynman diagram techniques. In such an approach the extraction of the operator anomalous dimension is relegated to the evaluation of the divergent part of a 2-point function derived from the parent 3-point one, $\langle A_\mu^a \mathcal{O} B_{\nu\sigma}^b \rangle$, where the external momentum of the $B_{\nu\sigma}^b$ field is nullified. Such a process for this Green's function is infrared safe since no infrared divergent factors such as $1/(k^2)^2$ arise in a Feynman integral where k is an internal momentum.

Green's function	One loop	Two loop	Three loop	Total
$A_\mu^a A_\nu^b$	5	52	1279	1336
$c^a \bar{c}^b$	1	8	152	161
$\psi^{iI} \bar{\psi}^{jJ}$	1	8	152	161
$B_{\mu\nu}^a \bar{B}_{\sigma\rho}^b$	1	20	464	485
$H_{\mu\nu}^a \bar{H}_{\sigma\rho}^b$	1	20	464	485
$A_\mu^a \bar{B}_{\nu\sigma}^b B_{\rho\phi}^c$	7	166	5827	6000
$A_\mu^a \mathcal{O} \bar{B}_{\nu\sigma}^b$	5	131	6917	7053
Total	21	405	15255	15681

Table 1. Number of Feynman diagrams for each Green's function.

In the final part of this setup description we note that we have to be careful in ensuring the correctness of the final expression. Since the operator insertion is in a Green's function involving a localizing ghost, we require a strong check on the $B_{\mu\nu}^a$ renormalization constants. To ensure this we have also performed the three loop $\overline{\text{MS}}$ renormalization of the $A_\mu^a \bar{B}_{\nu\sigma}^b B_{\rho\phi}^c$ vertex itself. As in the original QCD Lagrangian, this Green's function will produce the three loop $\overline{\text{MS}}$ β -function of the gauge coupling, [7, 24, 25]. Again as this is a 3-point function we nullify the external momentum of the $B_{\rho\phi}^c$ field relegating it to a 2-point function whence it can be determined by the MINCER algorithm. In Table 1, we have listed the number of Feynman diagrams computed for the present article. Those for the gluon and ghost exceed the numbers for the corresponding original QCD calculations due to the presence of the localizing fields. The

numbers of graphs in Table 1 are deduced from the QGRAF package, [26], which is the starting point for each of the Green's functions. The QGRAF routine generates the Feynman diagrams electronically and these are then converted to FORM input notation prior to the application of the MINCER algorithm. One additional complication is the non-trivial task of extending the FORM group theory module to handle the group theory associated with the λ^{abcd} couplings subject to the symmetry and Jacobi properties of (2) and (3). In addition we have also used the property noted in [6] that

$$\lambda^{acde}\lambda^{bcde} = \frac{1}{N_A}\delta^{ab}\lambda^{cdpq}\lambda^{cdpq} \quad , \quad \lambda^{acde}\lambda^{bdce} = \frac{1}{N_A}\delta^{ab}\lambda^{cdpq}\lambda^{cpdq} \quad (6)$$

and the analogous extension to the products of three λ^{abcd} -tensors with two free indices, which follow from the fact that there is only one rank two isotropic tensor in a classical Lie group. Finally, we note that the propagators of the (massless) localizing ghosts are given in [20].

We now record our main results at three loops. First, we define the renormalization constants for the relevant fields and the operator as

$$B_o^{a\mu\nu} = \sqrt{Z_B} B^{a\mu\nu} \quad , \quad H_o^{a\mu\nu} = \sqrt{Z_H} H^{a\mu\nu} \quad , \quad \mathcal{O}_o = Z_{\mathcal{O}} \mathcal{O} \quad (7)$$

where the subscript $_o$ denotes the bare quantity. Then the respective anomalous dimensions are

$$\gamma_B(a, \lambda, \alpha) = \mu \frac{d}{d\mu} \ln Z_B \quad , \quad \gamma_H(a, \lambda, \alpha) = \mu \frac{d}{d\mu} \ln Z_H \quad , \quad \gamma_{\mathcal{O}}(a, \lambda) = \mu \frac{d}{d\mu} \ln Z_{\mathcal{O}} \quad (8)$$

where we note

$$\mu \frac{d}{d\mu} = \beta(a) \frac{\partial}{\partial a} + \beta_{\lambda}^{pqrs}(a, \lambda) \frac{\partial}{\partial \lambda^{pqrs}} + \alpha \gamma_{\alpha}(a, \alpha) \frac{\partial}{\partial \alpha} \quad (9)$$

with $\beta(a)$ the β -function of the gauge coupling $a = g^2/(16\pi^2)$ and $\gamma_{\alpha}(a, \alpha)$ is the anomalous dimension of the linear covariant gauge fixing parameter. We use the conventions of [27] for this and note that $\gamma_{\alpha}(a, \alpha) = -\gamma_A(a, \alpha)$ with the latter defined to be the gluon anomalous dimension. In the term involving the β -function of the quartic couplings, $\beta^{abcd}(a, \lambda)$, it is understood that the differentiation respects the symmetries of the λ^{abcd} -tensor couplings, (2). Given this we find the $\overline{\text{MS}}$ expression

$$\begin{aligned} \gamma_B(a, \lambda, \alpha) &= \gamma_H(a, \lambda, \alpha) \\ &= (\alpha - 3)C_A + \left[\left(\frac{1}{4}\alpha^2 + 2\alpha - \frac{61}{6} \right) C_A^2 + \frac{10}{3} T_F N_f C_A \right] a^2 + \frac{1}{128 N_A} \lambda^{abcd} \lambda^{acbd} \\ &\quad + \left[\left(\frac{5}{16}\alpha^3 + \frac{39}{32}\alpha^2 + \frac{271}{32}\alpha - \frac{18193}{432} + \left(\frac{3}{8}\alpha^2 - \frac{27}{8} \right) \zeta(3) \right) C_A^3 \right. \\ &\quad \left. + \left(\frac{5}{54} + 48\zeta(3) - \frac{17}{4}\alpha \right) T_F N_f C_A^2 + (45 - 48\zeta(3)) T_F N_f C_F C_A \right. \\ &\quad \left. + \frac{140}{27} T_F^2 N_f^2 C_A \right] a^3 + \left[\frac{3}{8}\zeta(3) - \frac{13}{64} \right] \frac{C_A}{N_A} f_4^{abcd} \lambda^{acbd} a^2 \\ &\quad + \frac{1}{N_A} \left[\frac{13}{16} - \frac{3}{2}\zeta(3) \right] f_4^{abcd} f_4^{apcq} \lambda^{bpdq} a^2 + \frac{5C_A}{64N_A} \lambda^{abcd} \lambda^{acbd} a \\ &\quad - \frac{1}{2048N_A} \left[3\lambda^{abcd} \lambda^{acpq} \lambda^{bpdq} + \lambda^{abcd} \lambda^{apcq} \lambda^{bqdp} \right] + O(a^4; \lambda^4) \end{aligned} \quad (10)$$

where $\zeta(n)$ is the Riemann zeta function and a factor of $1/(4\pi)$, which derives from the loop integral measure, has formally been absorbed into each λ^{abcd} to simplify the presentation. The result (10) explicitly verifies the equality of the Slavnov-Taylor identity of [5, 6] to three loops.

The group Casimirs are defined by $\text{Tr}(T^a T^b) = T_F \delta^{ab}$, $T^a T^a = C_F$ and $f^{acd} f^{bcd} = C_A \delta^{ab}$. We have introduced the shorthand notation for the contraction of two structure functions

$$f_4^{abcd} = f^{eab} f^{ecd} \quad (11)$$

and defined the order symbol, $O(a^4; \lambda^4)$, to correspond to the four loop corrections. Moreover, with these values the three loop $\overline{\text{MS}}$ QCD β -function of [7, 24, 25] correctly emerges as λ^{abcd} and α independent from the $A_\mu^a \bar{B}_{\nu\sigma}^b B_{\rho\phi}^c$ vertex. Given this we find the three loop correction to the gluon mass operator is

$$\begin{aligned} \gamma_{\mathcal{O}}(a, \lambda) = & \left[\frac{11}{6} C_A - \frac{2}{3} T_F N_f \right] a + \left[\frac{77}{24} C_A^2 - \frac{2}{3} T_F N_f C_A - 2 T_F N_f C_F \right] a^2 \\ & - \frac{1}{16 N_A} f_4^{abcd} \lambda^{acbd} a - \frac{1}{256 N_A} \lambda^{abcd} \lambda^{acbd} \\ & + \left[\frac{361}{32} C_A^3 - \frac{211}{36} T_F N_f C_A^2 - \frac{97}{18} T_F N_f C_F C_A + T_F N_f C_F^2 \right. \\ & \quad \left. + \frac{5}{9} T_F^2 N_f^2 C_A + \frac{22}{9} T_F^2 N_f^2 C_F \right] a^3 + \frac{19}{32 N_A} f_4^{abcd} f_4^{apcq} \lambda^{bpdq} a^2 \\ & - \frac{1}{N_A} \left[\frac{1}{144} T_F N_f + \frac{857}{1152} C_A \right] f_4^{abcd} \lambda^{acbd} a^2 - \frac{19 C_A}{512 N_A} \lambda^{abcd} \lambda^{acbd} a \\ & + \frac{1}{N_A} \left[\frac{31}{768} f_4^{abcd} \lambda^{apcq} \lambda^{bdpq} + \frac{9}{512} f_4^{abcd} \lambda^{apbq} \lambda^{cpdq} - \frac{25}{768} f_4^{abcd} \lambda^{acpq} \lambda^{bpdq} \right] a \\ & + \frac{1}{4096 N_A} \left[3 \lambda^{abcd} \lambda^{acpq} \lambda^{bpdq} + \lambda^{abcd} \lambda^{apcq} \lambda^{bqdp} \right] + O(a^4; \lambda^4) \end{aligned} \quad (12)$$

which is clearly α independent as expected on general grounds but which in fact provides a non-trivial check on our computation. It is worth stressing that the emergence of the correct λ^{abcd} independent β -function and the gauge parameter independent $\gamma_{\mathcal{O}}(a, \lambda)$ is a non-trivial check on the implementation of the symmetry properties of λ^{abcd} in the FORM group theory module.

One additional calculational detail is worth noting and that is that λ^{abcd} itself undergoes a renormalization within the three loop calculations. Its *one* loop β -function was given in [6] as

$$\begin{aligned} \beta_{\lambda}^{abcd}(a, \lambda) = & \frac{1}{2} (d-4) \lambda^{abcd} + \frac{1}{8} \left[\lambda^{abpq} \lambda^{cqdp} + \lambda^{apbq} \lambda^{cdpq} + \lambda^{apcq} \lambda^{bpdq} + \lambda^{apdq} \lambda^{bpcq} \right] \\ & - 6 C_A \lambda^{abcd} a - 12 C_A f_4^{abcd} a^2 + 48 f_4^{apbq} f_4^{cpdq} a^2 + O(a^3; \lambda^3). \end{aligned} \quad (13)$$

Since we are going one loop beyond [6], it might have been expected that the two loop $\overline{\text{MS}}$ correction of (13) was needed. However, for the operator renormalization the first place λ^{abcd} occurs is at two loops. Therefore, one only needs its one loop renormalization. Equally for the $B_{\mu\nu}^a$ and $H_{\mu\nu}^a$ anomalous dimensions λ^{abcd} first appears at two loops and again only its one loop renormalization is necessary to deduce the fully renormalized three loop 2-point function. Here this is because the one loop graph involving λ^{abcd} which contributes to either 2-point function results in a snail graph which clearly is zero for the massless fields we consider. In other words it would only contribute to the renormalization of the $B_{\mu\nu}^a$ or $H_{\mu\nu}^a$ mass renormalization. Such a property of the λ^{abcd} -structure of the anomalous dimensions in fact prevents us from having to extend the λ^{abcd} renormalization to two loops by renormalizing massless 4-point functions which have non-safe nullifiable external momenta and hence not accessible to the MINCER algorithm. Moreover, it is worth noting that this is the first use of (13) within a computation and the overall consistency of our three loop renormalization is a non-trivial check on its correctness.

As a final check on our anomalous dimensions, we note that in the original renormalization constants we have been careful to check that the triple and double poles in ϵ are correctly

predicted from the known one and two loop structure. For the current Lagrangian, (1), this has an additional feature and that is that one has to take into account two coupling constants, a and λ^{abcd} . To aid the interested reader in this respect, we provide the explicit three loop $\overline{\text{MS}}$ renormalization constant for \mathcal{O} whence (12) was deduced. It is¹

$$\begin{aligned}
Z_{\mathcal{O}} = & 1 + \left[\frac{2}{3} T_F N_f - \frac{11}{6} C_A \right] \frac{a}{\epsilon} + \left[\frac{121}{24} C_A^2 + \frac{2}{3} T_F^2 N_f^2 - \frac{11}{3} T_F N_f C_A \right] \frac{a^2}{\epsilon^2} \\
& + \left[\left(\frac{1}{3} T_F N_f C_A - \frac{77}{48} C_A^2 + T_F N_f C_F \right) a^2 + \frac{1}{512 N_A} \lambda^{abcd} \lambda^{acbd} + \frac{1}{32 N_A} f_4^{abcd} \lambda^{acbd} \right] \frac{1}{\epsilon} \\
& + \left[\frac{605}{36} T_F N_f C_A^2 - \frac{6655}{432} C_A^3 - \frac{55}{9} T_F^2 N_f^2 C_A + \frac{20}{27} T_F^3 N_f^3 \right] \frac{a^3}{\epsilon^3} \\
& + \left[\left(\frac{3989}{288} C_A^3 - \frac{757}{72} T_F N_f C_A^2 - \frac{121}{18} T_F N_f C_A C_F + 2 T_F^2 N_f^2 C_A + \frac{22}{9} T_F^2 N_f^2 C_F \right) a^3 \right. \\
& \quad + \frac{1}{6144 N_A} \left(3 \lambda^{abcd} \lambda^{acpq} \lambda^{bpdq} + \lambda^{abcd} \lambda^{apcq} \lambda^{bqdp} \right) \\
& \quad + \frac{1}{N_A} \left(\frac{1}{384} f_4^{abcd} \lambda^{acpq} \lambda^{bpdq} + \frac{1}{256} f_4^{abcd} \lambda^{apbq} \lambda^{cpdq} + \frac{1}{384} f_4^{abcd} \lambda^{apcq} \lambda^{bpdq} \right) a \\
& \quad - \frac{C_A}{N_A} \left(\frac{1}{384} \lambda^{abcd} \lambda^{abcd} + \frac{31}{3072} \lambda^{abcd} \lambda^{acbd} \right) a + \frac{T_F N_f}{768 N_A} \lambda^{abcd} \lambda^{acbd} a \\
& \quad \left. + \frac{1}{16 N_A} f_4^{abcd} f_4^{apcq} \lambda^{bqdp} a^2 - \frac{41 C_A}{288 N_A} f_4^{abcd} \lambda^{acbd} a^2 + \frac{5 T_F N_f}{144 N_A} f_4^{abcd} \lambda^{acbd} a^2 \right] \frac{1}{\epsilon^2} \\
& + \left[\left(\frac{211}{108} T_F N_f C_A^2 - \frac{361}{96} C_A^3 + \frac{97}{54} T_F N_f C_A C_F - \frac{5}{27} T_F^2 N_f^2 C_A - \frac{22}{27} T_F^2 N_f^2 C_F \right. \right. \\
& \quad \left. - \frac{1}{3} T_F N_f C_F^2 \right) a^3 - \frac{1}{12288 N_A} \left(3 \lambda^{abcd} \lambda^{acpq} \lambda^{bpdq} + \lambda^{abcd} \lambda^{apcq} \lambda^{bqdp} \right) \\
& \quad + \frac{1}{N_A} \left(\frac{5}{128} f_4^{abcd} \lambda^{acpq} \lambda^{bpdq} - \frac{31}{384} f_4^{abcd} \lambda^{acpq} \lambda^{bpdq} \right. \\
& \quad \left. + \frac{17}{512} f_4^{abcd} \lambda^{apbq} \lambda^{cpdq} + \frac{5}{128} f_4^{abcd} \lambda^{apcq} \lambda^{bpdq} \right) a \\
& \quad + \frac{C_A}{N_A} \left(\frac{79}{1536} \lambda^{abcd} \lambda^{acbd} - \frac{5}{128} \lambda^{abcd} \lambda^{abcd} \right) a - \frac{19}{96 N_A} f_4^{abcd} f_4^{apcq} \lambda^{bqdp} a^2 \\
& \quad \left. + \frac{515 C_A}{3456 N_A} f_4^{abcd} \lambda^{acbd} a^2 + \frac{T_F N_f}{432 N_A} f_4^{abcd} \lambda^{acbd} a^2 \right] \frac{1}{\epsilon} + O(a^4; \lambda^4). \tag{14}
\end{aligned}$$

We note that the extraction of all our renormalization constants followed the procedure discussed in [28] for automatic Feynman diagram computations where all the Green's functions are first determined as a function of the bare quantities, such as the coupling constant. The renormalized versions are then introduced by the standard rescaling and the remaining divergences in ϵ are absorbed into the appropriate renormalization constant associated with that particular Green's function. This procedure is straightforward to implement within the FORM routines.

To conclude we have provided the full three loop $\overline{\text{MS}}$ renormalization of all quantities in the localizing Lagrangian, (1), for a renormalizable gauge invariant non-local mass operator, except the quartic couplings. Whilst this has been a cumbersome task it will actually play a crucial role when one considers the corresponding renormalization of the full operator of $\min_{\{U\}} \int d^4x (A_\mu^a)^2$ beyond the one loop result of [10]. Although this is clearly non-renormalizable, it is not inconceivable that one could localize the operator order by order in perturbation theory with a finite set of localizing fields at each order. Indeed the one loop result of [10] proceeded

¹We note that attached to the version of this article which appears on the arXiv there is a FORM file which contains the results (10), (12) and (14).

under this assumption. Whilst the additional couplings are absent at that level it would be interesting to see the structure of the gauge parameter independent anomalous dimension which emerges and to study the role of extra quartic couplings play in any renormalization group evolution. Moreover, given the successful extraction of the *three* loop anomalous dimension, it now also opens up the possibility of computing the two loop effective potential of this gauge invariant operator to study its condensation properties.

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